Acknowledgment

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Frequencies of Annular Plate and Curved Beam Elements

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I. Introduction

THERE is a precedent for modeling horizontally-curved highway bridges as curved beams[1,2] even though such bridges are generally composite structures consisting of curved girders and plates or box sections. For relatively lightweight vehicles, one can visualize a guideway cross section with a width perhaps ten times the depth dimension, especially for a two-lane elevated span. For such sections, plate theory may seem more appropriate for dynamic analysis than beam theory. This concern motivated the calculation and comparison of the free vibration frequency spectra of curved beam and annular plate elements. Limits of both theories are discussed in terms of dimensionless frequency parameters.

II. Curved Plate Free Vibrations

Based on classical theory for a uniform plate of stiffness D, deflection w, and mass density per unit surface area ρ, the governing homogeneous equation is[3]

$$D\nabla^4 w + \rho w_{,tt} = 0 \tag{1}$$

where the Laplacian operator ∇^2 is written in terms of the polar coordinates (r,θ) of Fig. 1.

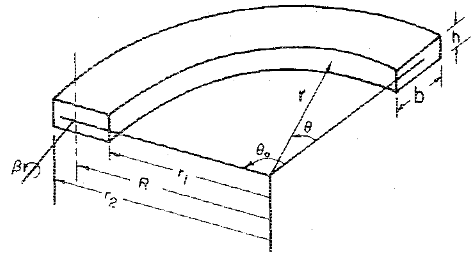

Fig. 1 Horizontally curved beam or plate segment.

Assuming this annular plate segment undergoes free harmonic vibrations at frequencies q, or

$$w(r,\theta,t) = W(r,\phi)\sin qt \tag{2}$$

then Eq. (1) reduces to

$$\nabla^2 W - k^4 W = 0 \qquad k^4 = \rho q^2/D \tag{3}$$

The mode solutions to Eq. (3) are chosen in the form

$$W(r,\theta) = \sum_{n=1}^{\infty} \Big[A_n J_n(kr) + B_n Y_n(kr) + C_n I_n(kr) + D_n K_n(kr)\Big] \sin(n\pi\theta/\theta_0) \tag{4}$$

where J_n and Y_n are Bessel functions of the first and second kind, respectively; I_n and K_n are modified Bessel functions of the first and second kind, respectively; and $A_n,\ldots,D_n$ are constants. Equation (4), with Eq. (2), satisfies both the zero deflection conditions at the end supports, $w(r,0,t) = w(r,\theta_0,t) = 0$ and the conditions of zero bending moment, $M_\theta(r,0,t) = M_\theta(r,\theta_0,t) = 0$. It remains to satisfy the conditions of zero radial moment and shear reaction at the two radial boundaries $r = r_i$ $(i = 1,2)$ or

$$M_r(r_i,\theta,t) = 0 \qquad V_r(r_i,\theta,t) = 0 \tag{5}$$

When Eqs. (5) are evaluated according to Eqs. (2) and (4), the result is four homogeneous, algebraic equations of the form

$$A[A_n B_n C_n D_n]^T = 0 \tag{6}$$

Nontrivial solutions to Eq. (6) exist only if the determinate of the coefficients vanishes, or

$$\det |A| = 0 \tag{7}$$

where the elements of the A matrix for $i = 1,2$ are

$$a_{ij} = S_n''(\alpha_i\lambda) + \frac{\nu}{\alpha_i\lambda} S_n'(\alpha_i\lambda) - \frac{\nu\beta_n^2}{(\alpha_i\lambda)^2} S_n(\alpha_i\lambda) \tag{8a}$$

$$a_{i+2,j} = S_n'''(\alpha_i\lambda) + \frac{1}{\alpha_i\lambda} S_n''(\alpha_i\lambda) - \frac{1}{(\alpha_i\lambda)^2} S_n'(\alpha_i\lambda) + \frac{2\beta_n}{(\alpha_i\lambda)^3} S(\alpha_i\lambda) + \frac{(1-\nu)}{(\alpha_i\lambda)^3}\beta_n^2 S(\alpha_i\lambda) - \frac{\beta_n^2}{(\alpha_i\lambda)^2} S_n'(\alpha_i\lambda) - \frac{(1-\nu)}{(\alpha_i\lambda)^2}\beta_n^2 S_n'(\alpha_i\lambda) \tag{8b}$$

In Eqs. (8), the symbol S_n is interpreted by J_n, Y_n, I_n, and K_n for $j = 1,2,3,4$, respectively. For convenience, the arguments (kr_i) of the Bessel functions were replaced by separate

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dimensionless forms $(\alpha_i\lambda)$ defined by

$$\alpha_1 = (1-\eta/2) \qquad \alpha_2 = (1+\eta/2) \tag{9}$$

where two dimensionless system parameters are now identified, using Eq. (3), as

$$\eta = b/R \qquad \lambda^2 = \lambda^2_{mn} = q_{mn} R^2 (\rho/D)^{1/2} \tag{10}$$

The procedure for calculating $\lambda = \lambda_{mn}$, and thus q_{mn}, from Eq. (10) is straightforward but tedious. By specifying η, the width to radius ratio, α_i is calculated from Eq. (9). Then, by fixing the index n (circumferential bending mode), consecutive values of λ_{mn}, $m = 1, 2, ...,$ are the consecutive roots of Eq. (7). Classical recursion formulas[4] were used to evaluate the derivatives of the Bessel functions in Eqs. (8), and a direct search routine, with an IBM 370/165 digital computer, was used to evaluate λ_{mn} for $0<\eta<1$. Numerical results will be discussed in Sec. IV.

III. Curved Beam Free Vibrations

In curved beam theory, the deflection $w = w(\theta,t)$ of the centroid of the cross section and the rotation $\beta = \beta(\theta,t)$ of this cross section are the dependent coordinates (Fig. 1). Previous results[5] for the curved beam frequencies p_{mn} are now recast for convenient comparison, with the results for plate theory just obtained.‡ The reference frequency is chosen as p, the fundamental frequency of a straight, simply supported beam of length l, or

$$p = \pi^2 (EI/m_0 l^4)^{1/2} \tag{11}$$

where EI is the stiffness and m_0 is the mass per unit length. The frequencies of the curved beam are given by

$$p_{mn} = n^2 p k^{1/2}_{mn} \qquad m = 1,2 \text{ only}; \qquad n = 1,2,3,... \tag{12}$$

where

$$k_{mn} = \frac{1}{2}\left[\xi^2\Theta_n^4\left(1+\frac{A}{\Theta_n^2}\right) + (1+A\Theta_n^2)\right] + \frac{(-1)^m}{2}\left\{\left[\xi^2\Theta_n^4\left(1+\frac{A}{\Theta_n^2}\right)+(1+A\Theta_n^2)\right]^2 - 4A\xi^2\Theta_n^2(1-\Theta_n^2)^2\right\}^{1/2} \tag{13}$$

$$\Theta_n = \theta_0/n\pi \qquad \xi = R/R_g \qquad A = GJ/EI \tag{14}$$

where $R_g = (I_p/A_c)^{1/2}$, the radius of gyration based on I_p, and the second moment of the cross section of area A_c about its polar axis. The rigidity ratio A is based only on the torsional stiffness GJ, a valid assumption for closed, box-type cross sections or solid, rectangular cross sections in which warping is negligible.[6]

For comparison of p_{mn} to q_{mn} of plate theory, a curved beam of rectangular dimensions, shown in Fig. 1, is assumed. Thus,

$$\xi = R\left[\frac{12(bh)}{bh^3+hb^3}\right]^{1/2} = \frac{\sqrt{12}}{\eta[1+(h/b)^2]^{1/2}} \simeq \frac{\sqrt{12}}{\eta} \tag{15}$$

where the square of the height to width ratio is ignored in comparison to unity. The rigidity ratio, based on $J = bh^3/3$ for relatively thin, rectangular cross sections,[7] becomes simply

$$A = \frac{GJ}{EI} = \frac{E}{2(1+\nu)}\cdot\frac{bh^3}{3}\cdot\frac{12}{Ebh^3} = \frac{2}{1+\nu} \tag{16}$$

‡The interpretation of subscripts on beam frequencies is reversed in Ref. 5.

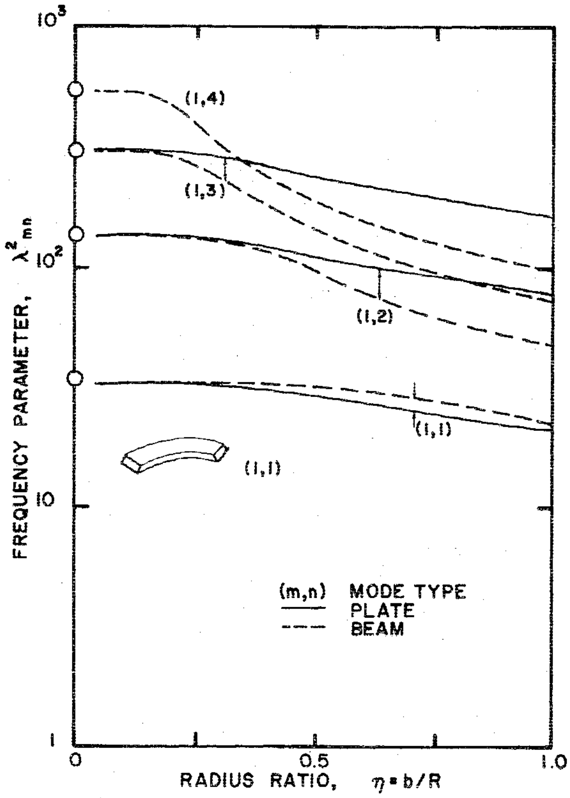

Fig. 2 Dominant bending frequencies for $\theta_0 = 30$ deg, $\nu = 0.3$.

Now, by rewriting Eq. (11) in terms of the curved plate parameters D, ρ, and arc length $R\theta_0 = l$, Eq. (12) assumes the form of its counterpart, Eq. (10), or

$$\lambda^2_{mn} = p_{mn} R^2 (\rho/D)^{1/2} = n^2\pi^2(1-\nu^2)^{1/2} k^{1/2}_{mn}/\theta_0^2 \tag{17}$$

Thus, for a fixed value of Poisson's ratio, A is known from Eq. (16), and λ^2_{mn} can be calculated from the right side of Eq. (17) for a fixed radius ratio $\eta = \sqrt{12}/\xi$ and arc angle θ_0.

IV. Element Frequencies

Figures 2 and 3 show typical numerical results for the frequency parameter λ^2_{mn} for several mode types (m,n) over a practical range of η, where $\theta_0 = 30$ deg and for Poisson's ratio, $\nu = 0.3$. In Fig. 2, the dominant circumferential bending modes $(n = 1,2,3,4,...)$ are each coupled with a clockwise twist of the cross sections $(m = 1)$. For $n = 1$, the central arc bends as a half-sine wave, the mode depicted in Fig. 2. Comparing these frequency parameters λ^2_{11}, λ^2_{12},... with corresponding values of λ^2_{21}, λ^2_{22} of Fig. 3 at the same η, the counterclockwise twist mode $(m = 2)$ yields frequencies about two orders of magnitude higher. Furthermore, the higher order twisting modes $m = 3$, 4,...corresponding to increasingly complex reversals in warping of the cross section, are predicted by plate theory, but not by beam theory. Since the latter plate frequencies corresponding to λ^2_{31}, λ^2_{32},..., λ^2_{41},... are again an order of magnitude higher than those for λ^2_{21}, λ^2_{22},..., these modes for $m \geq 3$ can probably be safely ignored in dynamic analysis for $\eta < 0.2$. In addition, since the solid lines for plate theory and the broken lines for beam theory merge for $\eta < 0.2$, it is concluded that curved beam theory is probably adequate for dynamic design analysis of curved elements in this range of η. A similar merging of frequencies was also observed for $\eta < 0.2$, where $\theta_0 = 60$ and 90 deg (Ref. 8).

V. Conclusions

The important conclusions are summarized. First, the circles at $\eta = 0$ in Fig. 2 correspond to reference bending

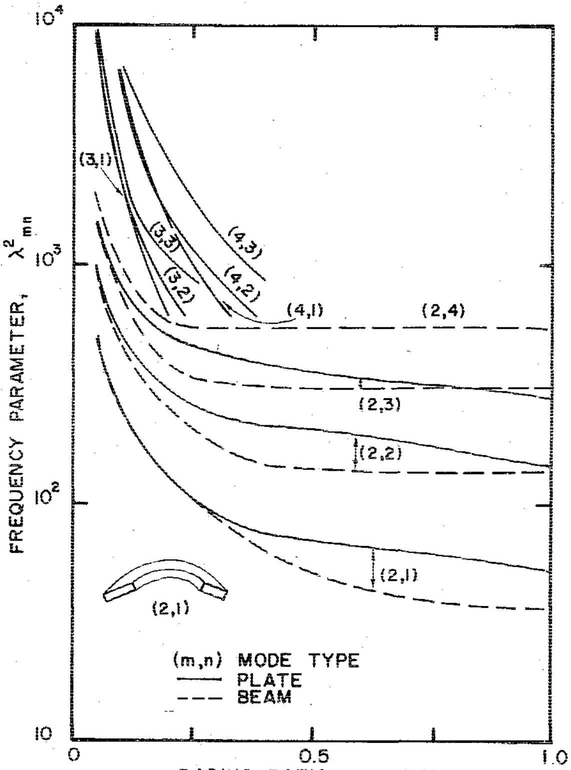

Fig. 3 Dominant twisting frequencies for $\theta_0 = 30$ deg, $\nu = 0.3$.

frequencies p [Eq. (11)] for a straight beam of the same length and stiffness as the respective curved segment. Corresponding curved element frequencies show increasing depression from these reference values as both η and θ_0 increase. Second, the fundamental frequency for plates is always lower than for beams, other parameters remaining the same. Third, although frequencies for $m \geq 3$ for $n = 1,2,...$ may be significant for $\eta > 0.2$, such frequencies are very high by comparison to those for $m = 1,2$ and $\eta > 0.2$. This implies that, for the low range of η, the curved beam theory can be employed with reasonable confidence in the dynamic design of curved elements. This is because there is little difference between the natural vibration frequencies calculated from plate theory and curved beam theory for ratios $\eta = b/R < 0.2$.

Acknowledgments

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Condensation of Free Body Mass Matrices Using Flexibility Coefficients

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REDUCTION of mass matrices for vibration analysis has become a routine part of many structural analysis systems. NASTRAN[1] includes a procedure by Guyan.[2] Kaufman and Hall[3] and Ramsden and Stoker[4] have suggested a procedure using flexibility coefficients. The procedure by Kaufman and Hall and Ramsden and Stoker is, however, limited to a restrained structure such that the flexibility matrix E is equal to the inverse of the stiffness matrix K.

The Guyan reduction is based on partitioning the stiffness matrix according to the deflection subsets U_s and U_o such that

$$\begin{bmatrix} K_{ss}K_{so} \\ K_{os}K_{oo} \end{bmatrix} \begin{Bmatrix} U_s \\ U_o \end{Bmatrix} = \begin{Bmatrix} P_s \\ O \end{Bmatrix} \qquad (1)$$

where U_s are the retained d.o.f. (degrees of freedom) and U_o are the omitted d.o.f. This leads to a coordinate transformation of the form

$$\begin{Bmatrix} U_s \\ U_o \end{Bmatrix} = \begin{bmatrix} I \\ -K_{oo}^{-1}\, K_{os} \end{bmatrix} \{U_s\} \qquad (2)$$

The method suggested by Kaufman and Ramsden and Stoker can be shown to be equivalent, if the flexibility matrix E is equal to K^{-1} such that

$$\begin{bmatrix} K_{ss}\ K_{so} \\ K_{os}\ K_{oo} \end{bmatrix} \begin{bmatrix} E_{ss}\ E_{so} \\ E_{os}\ E_{oo} \end{bmatrix} = \begin{bmatrix} I\ O \\ O\ I \end{bmatrix} \qquad (3)$$

Expansion to obtain the off-diagonal terms leads to

$$K_{os}E_{ss} + K_{oo}E_{os} = 0$$

Therefore

$$-K_{oo}^{-1}\, K_{os} = E_{os}E_{ss}^{-1}$$

The resulting transformation is

$$\begin{Bmatrix} U_s \\ U_o \end{Bmatrix} = \begin{bmatrix} I \\ E_{os}\, E_{ss}^{-1} \end{bmatrix} \{U_s\} \qquad (4)$$

For a free body the stiffness matrix is singular and K^{-1} does not exist. In this case we must define a subset of rigid body d.o.f. U_r and partition K such that

$$\left[\begin{array}{cc|c} K_{ss} & K_{sr} & K_{so} \\ K_{rs} & K_{rr} & K_{ro} \\ \hline K_{os} & K_{or} & K_{oo} \end{array}\right] \left\{\begin{array}{c} U_s \\ U_r \\ \hline U_o \end{array}\right\} = \left\{\begin{array}{c} P_s \\ P_r \\ \hline O \end{array}\right\} \qquad (5)$$

The reduction transformation is

$$\begin{Bmatrix} U_s \\ U_r \\ U_o \end{Bmatrix} = \left[\begin{array}{c|c} I & O \\ O & I \\ -K_{oo}^{-1}\, K_{os} & -K_{oo}^{-1}\, K_{or} \end{array}\right] \begin{Bmatrix} U_s \\ U_r \end{Bmatrix} \qquad (6)$$

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